On Relating Type Theories and Set Theories

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University of Leeds

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Brouwer and Bishop

- Luitzen Egbertus Jan Brouwer (1881-1966)

  

  Dissertation 1907
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  Zweiter Teil: Theorie der Punktmengen, 1919.

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- S. Feferman, Explicit Mathematics
- J. Myhill, Constructive Set Theory, **CST**.
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  CZF is a simplification and extension of Myhill's **CST**, induced by **MLTT**.
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Intuitionistic Zermelo-Fraenkel set theory, **IZF**

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Full Separation**
- **Powerset**
- **Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \quad \vdash \quad \exists b\ (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)$$

- **Set Induction**

$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x) \rightarrow\ \varphi(a)) \rightarrow \forall a\ \varphi(a),$$

- **IZF** has the same strength as **ZF** (Friedman).

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Myhill wanted to single out the principles that undergird Bishop's conceptions of what sets and functions are, adding that he wanted "these principles to be such as to make the process of formalization completely trivial, as it is in the classical case".

CST is based on intuitionistic logic

Many sorted system: numbers, sets, functions

Axioms (simplified)

- **Extensionality**
- **Pairing**, **Union**, **Infinity** (or $\mathbb{N}$ is a set)
- **Bounded Separation**
- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.
- **Replacement**
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Three papers

- P. Aczel: *The type theoretic interpretation of constructive set theory.* In: A. MacIntyre, L. Pacholski, J. Paris (eds.), *Logic Colloquium '77*, North Holland, Amsterdam (1978) 55–66.
- P. Aczel: *The type theoretic interpretation of constructive set theory: Choice principles.* In: A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, North Holland, Amsterdam (1982) 1–40.
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- **Extensionality**
- **Pairing**, **Union**, **Infinity**
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- **Subset Collection**

  For all sets A, B there exists a "sufficiently large" set of multi-valued functions from A to B.

- **Strong Collection**

$$(\forall x \in a)\, \exists y\; \varphi(x, y) \;\rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\; \varphi(x, y) \;\wedge\; (\forall y \in b)\, (\exists x \in a)\; \varphi(x, y)\,]$$

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Mathematics in **CZF** and Explosion

Mathematics can be formalized and developed in **CZF** in the usual way, e.g., using class notation, defining functions as just special relations.

Martin-Löf (1984) writes that:

> "The reason that B^A can be constructed as a set is that we take the notion of function as primitive, instead of defining a function as a set of ordered pairs or a binary relation satisfying the usual existence and uniqueness conditions, which would make it a category (like $\mathcal{P}(A)$) instead of a set."

CZF is proof-theoretically a weak theory but adding classical logic means **explosion**:

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Peter Aczel's interpretation in **MLTT**: New Axioms

- **Subset Collection**

  Given any sets A and B, there is a set C consisting of total relations from A to B such that for any total relation R from A to B, there exists $S \in C$ with $S \subseteq R$.

- **Presentation Axiom**.

  For any set A there exists a set B such that A is the surjective image of B and the axiom of choice holds over B.

- The $\Pi\Sigma$ and $\Pi\Sigma W$ **Axioms of Choice**.

- The **Regular Extension Axiom**.

  Every set is a subset of a regular set.

  A set C is said to be regular if it is transitive and inhabited and for every total relation R from a set $a \in C$ to C there exists a set $v \in C$ such that

$$\forall x \in a \exists y \in v\; xRy \;\wedge\; \forall y \in v \exists x \in a\; xRy.$$

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  Given any sets A and B, there is a set C consisting of total relations from A to B such that for any total relation R from A to B, there exists $S \in C$ with $S \subseteq R$.

- **Presentation Axiom**.

  For any set A there exists a set B such that A is the surjective image of B and the axiom of choice holds over B.

- The $\Pi\Sigma$ and $\Pi\Sigma W$ **Axioms of Choice**.

- The **Regular Extension Axiom**.

  Every set is a subset of a regular set.

  A set C is said to be regular if it is transitive and inhabited and for every total relation R from a set $a \in C$ to C there exists a set $v \in C$ such that

$$\forall x \in a \, \exists y \in v \; xRy \;\wedge\; \forall y \in v \, \exists x \in a \; xRy.$$

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The Type Theory $\mathbf{MLTT_1V}$

- In addition to the basic types, $\mathbf{MLTT_1V}$ has one universe $\mathcal{U}$ and the inductive type V of well-founded trees over $\mathcal{U}$:

$$\frac{A : \mathcal{U} \quad f : A \to \mathbf{V}}{\mathrm{sup}(A, f) : \mathrm{V}}$$

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Set-theoretic elementhood $\dot{\in}$ and equality $\doteq$ on $\mathbf{V}$

Every $\alpha : \mathbf{V}$ is of the form $\sup(A, f)$.

Writing $\{f(i) \mid i : A\}$ for $\sup(A, f)$ and $\{g(j) \mid j : B\}$ for $\sup(B, g)$, define $\dot{\in}$ and $\doteq$ on $\mathbf{V}$ inductively via

$$\{f(i) \mid i : A\} \doteq \{g(j) \mid j : B\} \quad \text{iff} \quad \begin{array}{l} \forall i : A\, \exists j : B\; f(i) \doteq g(j) \quad \wedge \\ \forall j : B\, \exists i : A\; g(j) \doteq f(i) \end{array}$$

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The strength of **MLTT**$_1$V

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as proof-theoretic ordinal:

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Theorem Let ψ be a mathematical statement and θ be a generalized mathematical statement expressed in set theory. The the following hold:

(i) $\mathbf{CZF} + \Pi\Sigma\text{-AC} \vdash \psi$ iff ψ is validated in $\mathbf{MLTT}_1^e\mathrm{V}$.

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Interpreting **CZF** in HoTT

Cesare Gallozzi (2018) investigated interpretations of **CZF** in HoTT.
He looked at a family of interpretations

$$[\![\cdot]\!]_k$$

for $2 \leq k \leq \infty$, where in the interpretation the indexing types from $\mathcal{U}$ have to be types of fixed homotopy level k.
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Peter Aczel's book Non-Well-Founded Sets 1988

Mathematical Conceptualism à la Weaver

Nick Weaver proposed a semi-intuitionistic theory **CM** of third-order arithmetic for axiomatizing what he calls mathematical conceptualism.

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The system **CM**

1. **CM** has
   - first order variables $n, m, k, \ldots$ (thought of as ranging over $\mathbb{N}$)
   - second order variables $X, Y, Z, \ldots$ (thought of as ranging over sets of naturals)
   - third order variables $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \ldots$ (thought of as ranging over sets of sets of naturals)
   - **Axioms**

     1.1 Number-theoretic axioms

     1.2 Law of excluded middle for formulas with no second or third order quantifiers.

     1.3 Induction on naturals for all formulas.

     1.4 Dependent choice at the second order level:
     If $\forall n \forall X \exists Y\, \psi(n, X, Y)$ then
     $\forall X \exists Z\, [Z_{(0)} = X \wedge \forall n\, \psi(n, Z_{(n)}, Z_{(n+1)})]$.

     1.5 Comprehension:
     $\forall n(\varphi(n) \vee \neg\varphi(n)) \rightarrow \exists X \forall n\, [n \in X \leftrightarrow \varphi(n)]$
     $\forall X(\vartheta(X) \vee \neg\vartheta(X)) \rightarrow \exists \mathbb{Y} \forall X\, [X \in \mathbb{Y} \leftrightarrow \vartheta(X)]$

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Developing mathematics in **CM**

It's actually quite easy.

1. The reals are a third order object, inhabited by Dedekind cuts of rationals.
2. A topological space is a set $\mathbb{X}$ together with a family of subsets $\mathfrak{T}$ of $\mathbb{X}$ such that (i) $\emptyset$ and $\mathbb{X}$ belong to $\mathfrak{T}$; (ii) the union of any sequence of sets that belong to $\mathfrak{T}$ belongs to $\mathfrak{T}$; and (iii) the intersection of any finitely many sets that belong to $\mathfrak{T}$ belongs to $\mathfrak{T}$.
3. Weaver shows that lot of topology, measure theory and functional analysis can be developed in **CM**. Core mathematics can be straightforwardly implemented in **CM**.

Principles of Omniscience

Limited Principle of Omniscience (LPO):

$$\forall f \in 2^{\mathbb{N}} \, [\exists n \, f(n) = 1 \quad \vee \quad \forall n \, f(n) = 0].$$

Lesser Limited Principle of Omniscience (LLPO):

$$\forall f \in 2^{\mathbb{N}} \big(\forall n, m[f(n) = f(m) = 1 \to n = m] \\ \to \; [\forall n \, f(2n) = 0 \quad \vee \quad \forall n \, f(2n+1) = 0]\big).$$

Dummett “Thought and Reality” 2006

- “If there are no gaps in reality, that is no questions that have no answers, then God’s logic will be classical.

  Those many people who favour classical over intuitionistic logic are therefore guilty of the presumption of reasoning as if they were God.”

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CM and Constructive Zermelo-Fraenkel set theory (**CZF**)

It is shown that it is unexpectedly easy to formalize a great deal of modern functional analysis in **CM**.

The interesting connection between **CZF** + LPO + RDC and **CM** is the following.

Theorem. **CM** can be interpreted in **CZF** + LPO + RDC.

Moreover, **CZF** + LPO + RDC is proof-theoretically reducible to **CZF** (R. 2014).

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The Anti-foundation Axiom

- A *graph* will consist of a set of *nodes* and a set of *edges*, each edge being an ordered pair $\langle x, y \rangle$ of nodes. If $\langle x, y \rangle$ is an edge then we will write $x \to y$ and say that y is a *child* of x.
- A *path* is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \ldots$ of nodes $x_0, x_1, x_2, \ldots$ linked by edges $\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots$.
- A *pointed graph* is a graph together with a distinguished node x_0 called its *point*. A pointed graph is *accessible* if for every node x there is a path $x_0 \to x_1 \to x_2 \to \ldots \to x$ from the point x_0 to x.
- A *decoration* of a graph is an assignment d of a set to each node of the graph in such a way that the elements of the set assigned to a node are the sets assigned to the children of that node, i.e.

$$d(a) = \{d(x) : \; a \to x\}.$$

- A *picture* of a set is an accessible pointed graph (apg for short) which has a decoration in which the set is assigned to the point.

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The *Anti-Foundation Axiom*, **AFA**, is the statement that every graph has a unique decoration.

Note that **AFA** has the consequence that every apg is a picture of a unique set.

AFA is in effect the conjunction of two statements:

- $\mathbf{AFA}_1$: *Every graph has at least one decoration.*
- $\mathbf{AFA}_2$: *Every graph has a most one decoration.*

$\mathbf{AFA}_1$ is an existence statement whereas $\mathbf{AFA}_2$ is a strengthening of the Extensionalty axiom of set theory. For example, taking the graph $\mathbb{G}_0$ to consist of a single node x_0 and one edge $x_0 \to x_0$, $\mathbf{AFA}_1$ ensures that this graph has a decoration $d_0(x) = \{d_0(y) : x \to y\} = \{d_0(x)\}$, giving rise to a set b such that $b = \{b\}$. However, if there is another set c satisfying $c = \{c\}$, the Extensionalty axiom does not force b to be equal to c, while $\mathbf{AFA}_2$ yields $b = c$. Thus, by **AFA** there is exactly one set Ω such that $\Omega = \{\Omega\}$.

Another example which demonstrates the extensionalizing effect of **AFA**$_2$ is provided by the graph $\mathbb{G}_\infty$ which consists of the infinitely many nodes x_i and the edges $x_i \to x_{i+1}$ for each $i \in \omega$. According to **AFA**$_1$, $\mathbb{G}_\infty$ has a decoration. As $d_\infty(x_i) = \Omega$ defines such a decoration, **AFA**$_2$ entails that this is the only one, whereby the different graphs $\mathbb{G}_0$ and $\mathbb{G}_\infty$ give rise to the same non-well-founded set.

The most important applications of **AFA** arise in connection with solving systems of equations of sets. In a nutshell, this is demonstrated by the following example. Let p and q be arbitrary fixed sets. Suppose we need sets x, y, z such that

$$\begin{aligned} x &= \{x, y\} \\ y &= \{p, q, y, z\} \\ z &= \{p, x, y\}. \end{aligned} \tag{1}$$

Here p and q are best viewed as atoms while x, y, z are the indeterminates of the system. **AFA** ensures that the system (1) has a unique solution. There is a powerful technique that can be used to show that systems of equations of a certain type have always unique solutions. In the terminology of Barwise, Moss "Vicious Circles" this is called the solution lemma.

The theory **CZFA**

CZFA is the theory **CZF**$\restriction$ plus **AFA** plus the axiom of transitive closure.

$\varepsilon_0 \leq |\mathbf{CZFA}| \leq \varphi 20$

Largest Fixed Points

Theorem: ($\mathbf{CZF}\restriction + \mathrm{RDC}$)

For every induction definition Φ*, there is a largest fixed point* $I^*(\Phi)$.
Indeed,

$$I^*(\Phi) \;=\; \bigcup\{x \mid x \text{ set and } x \subseteq \Gamma_\Phi(x)\}$$

$|\mathbf{CZF}\restriction + \mathrm{RDC}| = \varphi\varepsilon_0 0$

Some papers

- R. Lubarsky, M. Rathjen: *On the Constructive Dedekind Reals.* Logic and Analysis 1 (2008) 131-152.
- M. Rathjen: *The anti-foundation axiom in constructive set theories*, in: G. Mints, R. Muskens (eds.) *Games, Logic, and Constructive Sets.* (CSLI Publications, Stanford, 2003) 87–108.
- M. Rathjen: *Predicativity, circularity, and anti-foundation*. In: Godehard Link (ed.): *One Hundred Years of Russell's Paradox.* de Gruyter Series in Logic and its Applications 6 (Walter de Gruyter, Berlin, New York, 2004) 191–219.